

Self-similar interpolation in high-energy physics

V.I. Yukalov* and S. Gluzman

*Bogolubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna 141980, Russia*

Abstract

A method is suggested for interpolating between small-variable and large-variable asymptotic expansions. The method is based on self-similar approximation theory resulting in self-similar root approximants. The latter are more general than the two-sided Padé approximants and modified Padé approximants, including these as particular cases. Being more general, the self-similar root approximants guarantee the accuracy that is not worse, and often better, than that of the Padé approximants. The advantage of the root approximants is in their unambiguous definition and in the possibility of their construction, even when Padé approximants cannot be defined. Conditions for the unique definition of the root approximants are formulated. Several examples from high-energy physics illustrate the method.

PACS numbers: 02.30.Lt, 02.30.Mv, 02.60.Gf, 03.65.Db, 05.10.Cc, 11.10.Jj, 11.15.Tk, 11.25.Pm

***corresponding author:** V.I. Yukalov
E-mail: yukalov@theor.jinr.ru

1 Introduction

A very often met problem in high-energy physics is the necessity of constructing an analytical expression uniformly describing a function $f(x)$ in the whole interval of its domain, say, in $[0, \infty]$, when only asymptotic expansions are known for the small-variable limit $x \rightarrow 0$ and the large-variable limit $x \rightarrow \infty$. The variable x can represent, e.g., a coupling constant.

The standard way of treating such problems is the use of the so-called two-sided, or two-point, or multipoint Padé approximants [1]. In many cases, these approximants provide a reasonable interpolation between the small-variable and large-variable limits. However, this method has some weak points, as discussed in Refs. [1–3]. One of them is the occurrence of defects, such as spurious zeroes and poles. The other problem is the ambiguity in choosing one of the approximant variants from the Padé table. The important limitation is the requirement for the compatibility of expansions in the small-variable and large-variable limits. The latter means the following. The standard situation in many problems is when, in the small-variable limit, one has an expansion in integer powers, x^n , while the large-variable expansion exhibits the behavior x^β , with a noninteger power β . Since the large-variable behavior of a Padé approximant $P_{M/N}$ is x^{M-N} , this implies that the integer power $M - N$ is not compatible with the noninteger β . To overcome the problem of incompatibility, Baker and Gammel [4] suggested to use the fractional powers of Padé approximants $P_{M/N}^\gamma$, choosing the power γ so that $(M - N)\gamma = \beta$. The simplest case of the Baker-Gammel method is the polynomial approximant $P_{M/0}^\gamma$ in a fractional power $\gamma = \beta/M$. The Baker-Gammel method allows one to correctly represent the leading term of the large-variable behavior, although the subleading terms not always can be uniquely defined [5, 6].

In the present paper, we suggest an original method of interpolation between small-variable and large-variable expansions. This method allows one to construct an analytical expression uniformly approximating the sought function in the whole domain $[0, \infty]$ and reproducing both the small-variable and large-variable expansions. The uniqueness conditions are formulated allowing for a unique definition of all parameters. The method is more general than that of standard Padé approximants as well as the Baker-Gammel method of fractional Padé approximants. Therefore, by construction, the accuracy of our method is not worse, and often better, than that of the latter methods, with the advantage of being uniquely defined. The approach is illustrated by several examples from high-energy physics.

2 Method of self-similar interpolation

Let us be interested in a physical quantity represented by a real function $f(x)$ of a real variable $x \in [0, \infty]$. However, the explicit form of this function is not known, since it is defined by complicated equations allowing only for deriving asymptotic expansions in the vicinity of two ends, where $x \rightarrow 0$ and $x \rightarrow \infty$.

For instance, in the small-variable limit, we have

$$f(x) \simeq f_k(x) \quad (x \rightarrow 0) , \quad (1)$$

with the series

$$f_k(x) = f_0(x) \left(1 + \sum_{n=1}^k a_n x^n \right) , \quad (2)$$

where $f_0(x)$ is a known function. In many cases, the latter enjoys the form

$$f_0(x) = Ax^\alpha \quad (A \neq 0) , \quad (3)$$

with α being any real number.

And in the large-variable limit, we can get

$$f(x) \simeq f^{(p)}(x) \quad (x \rightarrow \infty) , \quad (4)$$

with the series

$$f^{(p)}(x) = \sum_{n=1}^p b_n x^{\beta_n} , \quad (5)$$

where the powers β_n are real numbers arranged in the descending order

$$\beta_{n+1} < \beta_n \quad (n = 1, 2, \dots, p-1) .$$

The values of the powers β_n can be any, either integer or fractional.

In what follows, it is convenient to deal with the reduced function $f(x)/f_0(x)$, normalized so that in the small-variable limit,

$$\frac{f(x)}{f_0(x)} \simeq \frac{f_k(x)}{f_0(x)} \quad (x \rightarrow 0) ,$$

we would have the simple asymptotic form

$$\frac{f_k(x)}{f_0(x)} = 1 + \sum_{n=1}^k a_n x^n .$$

The interpolation problem consists in constructing such a representation for the sought function $f(x)$ that would reproduce the small-variable, as well as large-variable expansions (2) and (5), providing an accurate approximation for the whole domain $[0, \infty]$.

It is worth stressing that this formulation of interpolation problem is rather general. In the majority of cases, practically all realistic problems can be reduced to this representation employing a change of variables. Also, the small-variable and large-variable limits are conditional, since it is always possible to interchange them by introducing the variable $t = 1/x$, or more generally, $t = 1/x^\mu$, with a positive μ .

The method, we suggest here, is based on self-similar approximation theory [7–11] combining the ideas of optimized perturbation theory, optimal control theory, dynamical theory, and renormalization-group approach. The main ideas of the theory are as follows. The transfer from an approximate form f_k to the form f_{k+1} is represented as a motion of a dynamical system in discrete time, whose role is played by the approximation order $k = 0, 1, 2, \dots$. The evolution equation of the dynamical system represents a kind of self-similar relation, where the name *self-similar approximation theory* comes from. The trajectory of the dynamical system, by construction, is bijective to the sequence $\{f_k\}$. The dynamical system in discrete time, that is, a cascade, can be embedded into a dynamical system in continuous time, that is, into a flow. The convergence of a sequence $\{f_k\}$ to its effective limit is equivalent to the convergence of the flow trajectory to a fixed point, which, in this way, corresponds to the

sought function f . The motion is governed by control functions guaranteeing fast convergence to the fixed point. The stability of the method is characterized by the map multipliers of the related dynamical system. We shall not repeat here all this machinery that has been expounded in all mathematical details in Refs. [7–11] and thoroughly described in review articles [12, 13], but we shall use the results of this approach.

Using the self-similar approximation theory for the purpose of interpolation between two asymptotic expansions, we come [13–16] to the *self-similar root approximant*

$$\frac{f_k^*(x)}{f_0(x)} = \left((\dots (1 + A_1 x)^{n_1} + A_2 x^2)^{n_2} + \dots + A_k x^k \right)^{n_k} . \quad (6)$$

First of all, we see that setting here all powers $n_j = \pm 1$, we can obtain different Padé approximants. And if the powers $n_j = \pm 1$, except the leading n_k that is found from the leading term of the large-variable expansion, then we get the modified Padé approximants of Baker and Gammel [4]. However, such a choice of the powers n_j is too restrictive and arbitrary. The parameters of approximants, to be uniquely defined, have to be prescribed by the available small-variable and large-variable expansions.

In our previous publications, the root approximants (6) were used so that the powers n_i and parameters A_i were defined through the one-sided expansion, say, the large-variable expansion, while the other expansion, e.g., small-variable expansion, was not reproduced. The attempts to find all values of n_i and A_i from the small-variable expansion resulted in the equations with multiple solutions. Such a nonunique definition of the parameters is, of course, unsatisfactory. Now, we aim at generalizing the use of the root approximants (6) in such a way that would allow us to uniquely define all powers n_i and parameters A_i and that both the small-variable as well as the large-variable expansions be reproduced.

We may notice that the root approximant (6) can be identically rewritten as

$$\frac{f_k^*(x)}{f_0(x)} = A_k^{n_k} x^{kn_k} \left(1 + \frac{B_{k-1}}{x^{m_{k-1}}} \left(1 + \frac{B_{k-2}}{x^{m_{k-2}}} \dots \frac{B_1}{x^{m_1}} \left(1 + \frac{1}{A_1 x} \right)^{n_1} \right)^{n_2} \dots \right)^{n_k} , \quad (7)$$

with the parameters

$$B_j = \frac{A_j^{n_j}}{A_{j+1}} \quad (j = 1, 2, \dots, k-1) \quad (8)$$

and powers

$$m_j = j + 1 - j n_j \quad (j = 1, 2, \dots, k-1) . \quad (9)$$

On the other hand, the large-variable expansion (5) can be represented in the form

$$\begin{aligned} f^{(p)}(x) = & b_1 x^{\beta_1} \left(1 + \frac{b_2}{b_1} x^{\beta_2 - \beta_1} \left(1 + \frac{b_3}{b_2} x^{\beta_3 - \beta_2} \dots \right. \right. \\ & \left. \left. \dots \frac{b_{p-1}}{b_{p-2}} x^{\beta_{p-1} - \beta_{p-2}} \left(1 + \frac{b_p}{b_{p-1}} x^{\beta_p - \beta_{p-1}} \right) \right) \dots \right) . \end{aligned} \quad (10)$$

Expanding the above expressions (7) and (10) in powers of $1/x$ and equating the similar terms, we find that these expansions are uniquely defined provided that Eq. (3) is valid,

$$A_k^{n_k} = \frac{b_1}{A} \quad (k = p) , \quad (11)$$

the largest power n_k is given by the relation

$$kn_k = \beta_1 - \alpha \quad (\alpha \neq \beta_1) , \quad (12)$$

and the other powers satisfy the equations

$$m_j = \beta_{k-j} - \beta_{k-j+1} \quad (j = 1, 2, \dots, k-1) . \quad (13)$$

In this way, all powers n_j of the root approximant (6) can be uniquely defined through the *uniqueness conditions*

$$jn_j = j + 1 - \beta_{k-j} + \beta_{k-j+1} \quad (j = 1, 2, \dots, k-1) . \quad (14)$$

It may happen that the number of terms in the small-variable and large-variable asymptotic expansions are not the same, $k \neq p$. Sometimes, just a single term of the large-variable expansion is known ($p = 1$), while several terms of the small-variable expansion are available. How then the uniqueness conditions (14) will be changed?

Let us assume that just the leading term of the large-variable behavior is known:

$$f(x) \simeq Bx^\beta \quad (x \rightarrow \infty) . \quad (15)$$

It is easy to notice that the general expansion (5) is reducible to the asymptotic form (15) by setting $\beta_n = \beta$. Then the uniqueness condition (14) reduces to the equality

$$n_j = \frac{j+1}{j} \quad (k > p = 1) , \quad (16)$$

where $j = 1, 2, \dots, k-1$, while the leading power reads as

$$n_k = \frac{\beta - \alpha}{k} \quad (k = 1, 2, \dots) . \quad (17)$$

All parameters A_i are uniquely defined from the small-variable expansion.

In the general case, we have k terms of the small-variable expansion and p terms of the large-variable expansion. Therefore, to satisfy these expansions, the root approximant must be of order $k + p$, possessing $k + p$ parameters A_j , among which k parameters A_j are defined by the accuracy-through-order procedure from the small-variable expansion and the remaining p parameters are defined from the large-variable expansion. But then we also need to have $k + p$ equations for determining $k + p$ powers n_j , while only p terms of the large-variable expansion are given. How all powers n_j could be found in such a case?

Fortunately, large-variable expansions practically always enjoy the following nice property. The difference

$$\Delta\beta_j \equiv \beta_j - \beta_{j+1} \quad (j = 1, 2, \dots, k-1) \quad (18)$$

between the nearest-neighbor powers is invariant:

$$\Delta\beta_j = \Delta\beta = \text{const} \quad (j = 1, 2, \dots, k-1) . \quad (19)$$

In that case, with the leading power being always given by an equation of type (12) and with all remaining powers n_j being defined by the uniqueness condition (14), we now have

$$n_{k+p} = \frac{\beta_1 - \alpha}{k+p} , \quad jn_j = j + 1 - \Delta\beta \quad (j = 1, 2, \dots, k+p-1) . \quad (20)$$

Then both the small-variable as well as the large-variable expansions can be satisfied, uniquely defining all parameters A_j , with $j = 1, 2, \dots, k+p$.

Below, we illustrate the method by several examples from high-energy physics.

3 Supersymmetric Yang-Mills circular Wilson loop

The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, in the limit of large number of colors N and strong t'Hooft coupling $\lambda = g^2 N$ is taken sometimes as a model for hot QCD [17]. Since there exists an exactly calculable expression for the $SU(N)$ circular Wilson loop [18, 19], it is useful to start with this case, for which the accuracy of approximations can be explicitly estimated.

The exact circular Wilson loop is given by

$$W(\lambda) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) , \quad (21)$$

where I_1 is a modified Bessel function of the first kind. In the weak-coupling limit, one has

$$W(\lambda) \simeq e^{\sqrt{\lambda}} \left(1 - \lambda^{1/2} + \frac{5}{8} \lambda - \frac{7}{24} \lambda^{3/2} + \frac{7}{64} \lambda^2 \right) \quad (\lambda \rightarrow 0) , \quad (22)$$

and in the strong coupling limit,

$$W(\lambda) \simeq e^{\sqrt{\lambda}} \left(\sqrt{\frac{2}{\pi}} \lambda^{-3/4} - \frac{3}{4\sqrt{2\pi}} \lambda^{-5/4} \right) \quad (\lambda \rightarrow \infty) . \quad (23)$$

Introducing the change of the variables as

$$f(x) \equiv W(\lambda(x)) , \quad \lambda = x^2 , \quad (24)$$

we obtain the weak-coupling limit

$$f(x) \simeq e^x \left(1 - x + \frac{5}{8} x^2 - \frac{7}{24} x^3 + \frac{7}{64} x^4 \right) \quad (x \rightarrow 0) \quad (25)$$

and the strong-coupling limit

$$f(x) \simeq e^x \left(\sqrt{\frac{2}{\pi}} x^{-3/2} - \frac{3}{4\sqrt{2\pi}} x^{-5/2} \right) \quad (x \rightarrow \infty) . \quad (26)$$

According to rule (20), we find

$$n_1 = n_2 = n_3 = n_4 = n_5 = 1 , \quad n_6 = -\frac{1}{4} . \quad (27)$$

Then the corresponding root approximant becomes

$$f_6^*(x) = e^x \left(1 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 + A_6 x^6 \right)^{-1/4} , \quad (28)$$

acquiring the form of a modified Padé approximant, with

$$\begin{aligned} A_1 &= 4 , & A_2 &= \frac{15}{2} , & A_3 &= \frac{26}{3} , \\ A_4 &= \frac{653}{96} , & A_5 &= \frac{3\pi^2}{8} , & A_6 &= \frac{\pi^2}{4} . \end{aligned}$$

The maximal deviation of the root approximant f_6^* from the function, corresponding to the exact Wilson loop W , is 0.003. This accuracy is the same as that of the Padé approximant $P_{5/7}$, studied in Ref. [20] and requiring the knowledge of twice more terms of the asymptotic expansions.

As has been mentioned above, small-variable and large-variable expansions can be interchanged by a change of variables. Thus, in the present case, we may use the change

$$\varphi(t) \equiv W(\lambda(t)) , \quad \lambda = \frac{1}{t^2} . \quad (29)$$

Then, dealing with the variable t and following the general rule, we get the root approximant

$$\varphi_6^*(t) = \sqrt{\frac{2}{\pi}} e^{1/t} t^{3/2} (1 + B_1 t + B_2 t^2 + B_3 t^3 + B_4 t^4 + B_5 t^5 + B_6 t^6)^{-1/4} , \quad (30)$$

with the parameters

$$\begin{aligned} B_1 &= \frac{3}{2} , & B_2 &= \frac{653}{24\pi^2} , & B_3 &= \frac{104}{3\pi^2} , \\ B_4 &= \frac{30}{\pi^2} , & B_5 &= \frac{16}{\pi^2} , & B_6 &= \frac{4}{\pi^2} . \end{aligned}$$

Approximants (28) and (30) coincide with each other.

This example is also interesting demonstrating how the modified Padé approximants naturally arise in our method. That is, it is shown that our method includes the modified Padé approximants as a particular case. Such a reduction, of course, happens not always, but rather rarely. The root approximants (6) enjoy a more general form than Padé approximants, because of which they can provide better accuracy.

4 Planar cusp anomalous dimension in supersymmetric Yang-Mills theory

In the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, in the limit of large angle, the planar cusp anomalous dimension is linear in angle, with a coefficient $\Gamma(g)$ that is the cusp anomalous dimension of a light-like Wilson loop, which depends only on the coupling g . The weak-coupling and strong-coupling expansions [21–26] are

$$\Gamma(g) \simeq 4g^2 - \frac{4\pi^2}{3} g^4 + \frac{44\pi^4}{45} g^6 - 8 \left[\frac{73\pi^2}{630} + 4\zeta^2(3) \right] g^8 \quad (g \rightarrow 0) \quad (31)$$

and, respectively,

$$\Gamma(g) \simeq 2g - \frac{3 \ln 2}{2\pi} \quad (g \rightarrow \infty) . \quad (32)$$

The corresponding root approximant reads as

$$\Gamma_5^*(g) = 4g^2 \left(\left(\left(\left((1 + A_1 g^2)^{3/2} + A_2 g^4 \right)^{5/4} + A_3 g^6 \right)^{7/6} + A_4 g^8 \right)^{9/8} + A_5 g^{10} \right)^{-1/10} , \quad (33)$$

where

$$A_1 = \frac{256\pi^2}{189}, \quad A_2 = \frac{13376\pi^4}{59535}, \quad A_3 = \frac{32}{6751269} [54091\pi^6 + 12859560\zeta^2(3)],$$

$$A_4 = 256 \left(\frac{15 \ln 2}{\pi} \right)^{8/9}, \quad A_5 = 1024.$$

This form (33) practically coincides with the Padé approximant $P_{5/6}$ given in Ref. [20], differing from it only by 1%.

5 Spinor mass in heterotic string theory

The $SO(32)$ spinor mass in heterotic string theory admits [5,20] a perturbative weak-coupling expansion

$$M(g) \simeq g^{1/4} (1 + 0.23g^2) \quad (g \rightarrow 0) \quad (34)$$

and a dual expansion in the limit of strong coupling,

$$M(g) \simeq g^{3/4} (1 + 0.351g^{-1}) \quad (g \rightarrow \infty). \quad (35)$$

The root approximant for this case is

$$M_3^*(g) = g^{1/4} \left(\left((1 + A_1 g^2)^{3/2} + A_2 g^4 \right)^{5/4} + A_3 g^6 \right)^{1/12}, \quad (36)$$

with

$$A_1 = 1.472, \quad A_2 = 3.159299, \quad A_3 = 1.$$

This form is very close to the Padé approximant $P_{4/1}$ calculated in Ref. [5], the maximal difference being only 0.5%.

6 Ground-state energy for Schwinger model

The Schwinger model [27, 28] is a lattice gauge theory in $(1+1)$ dimensions representing Euclidean quantum electrodynamics with a Dirac fermion field. It possesses many properties in common with QCD, such as confinement, chiral symmetry breaking, and charge shielding. For this reason, it has become a standard test bed for the study of numerical techniques.

Here we consider the ground state of the model, corresponding to a vector boson of mass $M(x)$ as a function of the variable $x = m/g$, where m is electron mass and g is the coupling parameter having the dimension of mass, so that x is dimensionless. The energy is given by the relation $E = M - 2m$. The small- x expansion for the ground-state energy [29–32] is

$$E(x) \simeq 0.5642 - 0.219x + 0.1907x^2 \quad (x \rightarrow 0). \quad (37)$$

In the large- x limit [32–35], we have

$$E(x) \simeq 0.6418x^{-1/3} - \frac{1}{\pi} x^{-1} - 0.25208x^{-5/3} \quad (x \rightarrow \infty). \quad (38)$$

The corresponding root approximant is

$$E_5^*(x) = A \left(\left(\left(\left((1 + A_1 x)^{4/3} + A_2 x^2 \right)^{7/6} + A_3 x^3 \right)^{10/9} + A_4 x^4 \right)^{13/12} + A_5 x^5 \right)^{-1/15}, \quad (39)$$

where

$$\begin{aligned} A &= 0.5642, & A_1 &= 3.109547, & A_2 &= 3.640565, \\ A_3 &= 4.028571, & A_4 &= 1.070477, & A_5 &= 0.144711. \end{aligned}$$

The accuracy of this approximant E_5^* can be compared to data obtained in other calculations, density matrix renormalization group, E_{DMRG} , by Byrnes et al. [36] and fast moving frame estimates, E_{FMFE} by Kröger and Scheu [37]. This energy has also been calculated by using variational perturbation theory [38], which however has been found to be rather complicated and having no advantage over other techniques, such as Padé approximants [35]. Adam [39] used renorm-ordered perturbation theory, although his results are less accurate than E_{DMRG} and E_{FMFE} . In Table 1, we compare the latter data with our result E_5^* . Also, we present the energy E_{PA} found by means of the Padé approximant $P_{5/6}(x^{1/3})$. All results are close to each other, being practically the same in the frame of calculational errors.

7 Prediction of large-variable expansions

The great advantage of the method of self-similar root approximants, as compared to Padé approximants, is that root approximants can predict the correct behavior of sought functions at large variables, being based on small-variable expansions. Such a prediction is principally impossible by means of Padé approximants, when the small-variable and large-variable expansions contain incompatible powers [1–3], for example, when the small-variable expansion is in integer powers, as in Eq. (2), while the large-variable expansion, as in Eq. (5), is in fractional powers.

To illustrate this basic advantage, let us consider the ground-state energy for the Schwinger model, studied in the previous section. Suppose, only the small-variable expansion (37) is available and the powers $-1/3$ and -1 of expansion (38) are known. But no coefficients from the large-variable expansion are given. With the two terms of expansion (37), we can construct the root approximant

$$E_2^*(x) = A \left((1 + A_1 x)^{4/3} + A_2 x^2 \right)^{-1/6}, \quad (40)$$

in which all parameters are found from the small-variable expansion (37):

$$A = 0.5642, \quad A_1 = 1.746721, \quad A_2 = 0.458024.$$

By extending this expression to large variables, we get

$$E_2^*(x) \simeq 0.642616x^{-1/3} \quad (x \rightarrow \infty). \quad (41)$$

As is seen, the value 0.6426 very well approximates the first coefficient 0.6418 of the exact expansion (38).

Moreover, taking into account this prediction (41), we can construct the root approximant

$$E_3^*(x) = A \left(\left((1 + A_1 x)^{4/3} + A_2 x^2 \right)^{7/6} + A_3 x^3 \right)^{-1/9}, \quad (42)$$

where again the parameters A , A_1 , and A_2 are defined by the small-variable expansion (37), resulting in

$$A = 0.5642, \quad A_1 = 2.245784, \quad A_2 = 1.336080, \quad A_3 = 0.309979.$$

At large variables, Eq. (42) yields

$$E_3^*(x) \simeq 0.642616x^{-1/3} - 0.322985x^{-1} \quad (x \rightarrow \infty). \quad (43)$$

Both coefficients here are close to those of the exact expansion (38). The value of the second coefficient 0.322985 has to be compared with $1/\pi = 0.318310$.

8 Remarks on scheme and scale invariance

It is worth noting that self-similar approximation theory [7–11], used for deriving self-similar root approximants, considered in the present paper, employs the ideas of renormalization group in a sense that is different from this notion in quantum field theory, although being close mathematically. In self-similar approximation theory, the transfer from one approximation of order k to another, say, $k + 1$, is considered as a motion in discrete time k . Then the sequence of approximations can be treated as a cascade. Embedding the cascade into a flow makes it possible to pass from discrete time k to continuous time t . The flow evolution is represented by the renormalization-group equation of the type $df(x)/dt = v(x)$, where $v(x)$ is a flow velocity. Solving the latter equation iteratively leads to self-similar approximants for the sought function $f(x)$.

In quantum field theory, such as QCD or QED, there is the known problem of scheme and scale dependence of truncated series. The sought function $f(x)$ is not scheme-scale dependent, while its truncated series, as in Eq. (2), generally, can be dependent on both. Thus, if $f(x)$ is an observable quantity that is a function of the coupling parameter $x = \alpha_s/\pi$, then both the coefficients a_n , as well as the coupling parameter x , can depend on renormalization scheme and scale. The dependence of x on scale comes from the renormalization-group equation $dx/dt = \beta(x)$, where t is a scale shift and $\beta(x)$ is a Gell-Mann-Low function. It has been shown that Padé approximants reduce the scale dependence, when the Gell-Mann-Low function is taken in one-loop approximation [40, 41], and a generalized approach [42–45] has been developed for achieving perturbative scale invariance for any number of loops.

In the cases we have considered, the situation is different from the mentioned quantum-field *extrapolation* problems, since we investigate the *interpolation* method, where both the small and large variable limits of the sought function are given. In our case, there can arise the question of invariance with respect to a parameter, if expansion (2) is not unique, which is analogous to the scheme dependence. For example, there can exist different perturbative schemes, associated with different values of a parameter t , so that the coefficients $a_n = a_n(t)$ in expansion (2) depend on this parameter, while the sought function $f(x)$ should not depend on such parameters. Then the truncated series $f_k(x, t)$ also depend on the parameter, as

a result of which the root approximant $f_k^*(x, t)$ includes the dependence on t . However, in the interpolation problem, the small-variable asymptotic form $f_0(x)$ is given, being not dependent on auxiliary parameters, as well as the large-variable limit is also assumed to be available, say, as the asymptotic behavior (15), where B does not depend on t . This implies that the small-variable limit

$$\lim_{x \rightarrow 0} \frac{f_k^*(x, t)}{f(x)} = \lim_{x \rightarrow 0} \frac{f_k^*(x, t)}{f_0(x)} = 1 \quad (44)$$

is fixed, together with the large-variable limit

$$\lim_{x \rightarrow \infty} \frac{f_k^*(x, t)}{f(x)} = \lim_{x \rightarrow \infty} \frac{f_k^*(x, t)}{Bx^\beta} = 1. \quad (45)$$

For instance, if the small-variable behavior is given by Eq. (3) and only the large-variable asymptotic form (15) is available, then the root approximant is

$$f_k^*(x, t) = Ax^\alpha \left((\dots (1 + A_1x)^2 + A_2x^2)^{3/2} + \dots + A_kx^k \right)^{n_k}, \quad (46)$$

with n_k defined in Eq. (17). The dependence on t comes from the quantities A_n that are expressed through the coefficients $a_n(t)$ in the process of the accuracy-through-order procedure. Then the large-variable limit (45) yields

$$A \left((\dots (A_1^2 + A_2^2)^{3/2} + A_3) + \dots + A_k \right)^{(\beta-\alpha)/k} = B. \quad (47)$$

Since the right-hand sides in Eqs. (44), (45), or (47) do not depend on t , the root approximant is *asymptotically* t -invariant.

Moreover, the root approximants, as has been shown, approximate well the sought function in the whole interval of the variable $x \in [0, \infty)$. More precisely, the root approximant $f_k^*(x, t)$ uniformly approximates the sought function $f(x)$, with the maximal error ε , so that

$$\left| \frac{f_k^*(x, t) - f(x)}{f(x)} \right| < \varepsilon. \quad (48)$$

It is easy to see that if there are two approximants, related to two different parameters t_1 and t_2 , such that

$$\left| \frac{f_k^*(x, t_1) - f(x)}{f(x)} \right| < \varepsilon_1, \quad \left| \frac{f_k^*(x, t_2) - f(x)}{f(x)} \right| < \varepsilon_2,$$

then the difference between these approximants is described by the inequality

$$\left| \frac{f_k^*(x, t_1) - f_k^*(x, t_2)}{f(x)} \right| < \varepsilon_1 + \varepsilon_2. \quad (49)$$

In that sense, the root approximants, within the given accuracy, are *approximately scheme invariant*, that is, invariant with respect to the parameter t labelling different perturbative schemes.

The dependence on a parameter can also occur in the change of the variables $x = x(z, t)$, similarly to the scale dependence in field theory. Then it is possible to expand $f_k(x(z, t))$ in powers of the new variable z , getting $g_k(z, t)$. The corresponding root approximant $g_k^*(z, t)$, after the inverse transformation $z = z(x, t)$, resulting in $g_k^*(z(x, t), t)$ is such that, by construction, it uniformly approximates the sought function $f(x)$, so that

$$\left| \frac{g_k^*(z(x, t), t) - f(x)}{f(x)} \right| < \varepsilon', \quad (50)$$

within the maximal error ε' . This allows us to classify the root approximants as *approximately scale invariant*, within the given accuracy.

We would like to note it again that the problem of interpolation is different from that of extrapolation. For the latter, the problem of scheme and scale invariance is more complicated. We plan to consider this in future publications.

9 Conclusion

We have suggested a general and simple method for interpolation between small-variable and large-variable asymptotic expansions. The method is based on self-similar approximation theory, which allows for the construction of approximations, whose form follows from extracting the properties of functional similarity between the given expansion orders. Mathematical details of this theory can be found in the cited references. The resulting self-similar root approximant (6) makes it possible to satisfy the small as well as large variable expansions and to uniformly describe the sought function in the whole domain of its definition.

The general form of the root approximant (6) includes as particular cases that of Padé approximants, because of which the accuracy of root approximants is not worse than that of Padé approximants. But the root approximants enjoy several advantages, as compared to Padé approximants.

First, a root approximant for given orders k and p is uniquely defined, while Padé approximants for each given orders, allow for multiple representations. Really, for a small-variable expansion of order k and large-variable expansion of order p , it is admissible to construct the whole table of Padé approximants $P_{M/N}$, with different M and N satisfying the equality $M + N = k + p + 1$. Also, when one intends to construct diagonal Padé approximants, one needs an even number of terms in a small-variable expansion, while root approximants can be formed for any number of such terms.

Second important advantage is that root approximants make it possible to predict large-variable expansions, being based on small-variable expansions. Such a prediction by means of Padé approximants is principally impossible, when the small-variable and large-variable expansions contain incompatible powers. Moreover, in the case of incompatible expansions, Padé approximants cannot be defined at all.

We have formulated uniqueness conditions allowing us to uniquely define all parameters of the root approximant (6) from the coefficients of the given asymptotic expansions. The root approximants are shown to be approximately scheme and scale invariant.

The use of the interpolation formula (6) is convenient for the problems of high energy physics, when, due to the duality between weak coupling and strong coupling, there exist

asymptotic expansions for both these limits. We have illustrated our approach by several examples, demonstrating the generality, simplicity, and good accuracy of this method.

Acknowledgement

One of the authors (V.I.Y.) is grateful to E.P. Yukalova for useful discussions.

References

- [1] G.A. Baker and P. Graves-Morris, *Padé Approximants* (Cambridge University, Cambridge, 1996).
- [2] G.A. Baker, Defects and the convergence of Padé approximants, *Acta Appl. Math.* **61**, 37–52 (2000).
- [3] S. Gluzman and V.I. Yukalov, Extrapolation of perturbation theory expansions by self-similar approximants, *Eur. J. Appl. Math.* **25**, 595–628 (2014).
- [4] G.A. Baker and J.L. Gammel, The Padé approximant, *J. Math. Anal. Appl.* **2**, 21–30 (1961).
- [5] A. Sen, S-duality improved superstring perturbation theory, *J. High En. Phys.* **11**, 029 (2013).
- [6] M. Honda, On perturbation theory improved by strong coupling expansion, *J. High En. Phys.* **12**, 019 (2014).
- [7] V.I. Yukalov, Statistical mechanics of strongly nonideal systems, *Phys. Rev. A* **42**, 3324–3334 (1990).
- [8] V.I. Yukalov, Self-similar approximations for strongly interacting systems, *Physica A* **167**, 833–860 (1990).
- [9] V.I. Yukalov, Method of self-similar approximations, *J. Math. Phys.* **32**, 1235–1239 (1991).
- [10] V.I. Yukalov, Stability conditions for method of self-similar approximations, *J. Math. Phys.* **33**, 3994–4001 (1992).
- [11] V.I. Yukalov and E.P. Yukalova, Temporal dynamics in perturbation theory, *Physica A* **225**, 336–362 (1996).
- [12] V.I. Yukalov and E.P. Yukalova, Self-similar perturbation theory, *Ann. Phys. (N.Y.)* **277**, 219–254 (1999).
- [13] V.I. Yukalov and E.P. Yukalova, Self-similar structures and fractal transforms in approximation theory, *Chaos Solit. Fract.* **14**, 839–861 (2002).
- [14] V.I. Yukalov, E.P. Yukalova, and S. Gluzman, Self-similar interpolation in quantum mechanics, *Phys. Rev. A* **58**, 96–115 (1998).
- [15] S. Gluzman and V.I. Yukalov, Unified approach to crossover phenomena, *Phys. Rev. E* **58**, 4187–4209 (1998).
- [16] V.I. Yukalov and S. Gluzman, Self-similar crossover in statistical physics, *Physica A* **273**, 401–415 (1999).
- [17] J.P. Blaizot, E. Iancu, U. Kraemmer, and A. Rebhan, Hard thermal loops and the entropy of supersymmetric Yang-Mills theories, *J. High En. Phys.* **6**, 035 (2007).

- [18] J.K. Erickson, G.W. Semenoff, and K. Zarembo, Wilson loop in $N = 4$ supersymmetric Yang-Mills theory, Nucl. Phys. B **582**, 155–175 (2000).
- [19] N. Drukker and D.J. Gross, An exact prediction of $N = 4$ SUSYM theory for string theory, J. Math. Phys. **42**, 2896–2914 (2001).
- [20] T. Banks and T.J. Torres, The two-point Padé approximants and duality, arXiv:1307.3689 (2013).
- [21] G.P. Korchemsky and A.V. Radyushkin, Renormalization of the Wilson loops beyond the leading order, Nucl. Phys. B **283**, 342–364 (1987).
- [22] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, A semiclassical limit of the gauge-string correspondence, Nucl. Phys. B **636**, 99–114 (2002).
- [23] S. Frolov and A.A. Tseytlin, Semiclassical quantization of rotating suprestring in $AdS_5 \times S_5$, J. High En. Phys. **0206**, 007 (2002).
- [24] A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko, and V.N. Velizhanin, Three loop universal anomalous dimension of the Wilson operators in $N = 4$ SUSY Yang-Mills model, Phys. Lett. B **595**, 521–529 (2004).
- [25] N. Beisert, B. Eden, and M. Staudacher, Transcendentality and crossing, J. Stat. Mech. **0701**, 01021 (2007).
- [26] D. Correa, J. Henn, J. Maldacena, and A. Sever, The cusp anomalous dimension at three loops and beyond, J. High En. Phys. **1205**, 098 (2012).
- [27] J. Schwinger, Gauge invariance and mass, Phys. Rev. **128**, 2425–2428 (1962).
- [28] T. Banks, L. Susskind, and J. Kogut, Strong-coupling calculations of lattice gauge theories: (1+1)-dimensional exercises, Phys. Rev. D **13**, 1043–1053 (1976).
- [29] A. Carrol, J. Kogut, D.K. Sinclair, and L. Susskind, Lattice gauge theory calculations in 1 + 1 dimensions and the approach to the continuum limit, Phys. Rev. D **13**, 2270–2277 (1976).
- [30] J.P. Vary, T.J. Fields, and H.J. Pirner, Chiral perturbation theory in the Schwinger model, Phys. Rev. D **53**, 7231–7238 (1996).
- [31] C. Adam, The Schwinger mass in the massive Schwinger model, Phys. Lett. B **382**, 383–388 (1996).
- [32] P. Striganesh, C.J. Hamer, and R.J. Bursill, A new finite-lattice study of the massive Schwinger model, Phys. Rev. D **62**, 034508 (2000).
- [33] S. Coleman, More about the massive Schwinger model, Ann. Phys. (N.Y.) **101**, 239–267 (1976).
- [34] C.J. Hamer, Lattice model calculations for $SU(2)$ Yang-Mills theory in 1 + 1 dimensions, Nucl. Phys. B **121**, 159–175 (1977).

- [35] C.J. Hamer, Z. Weihong, and J. Oitmaa, Series expansions for the massive Schwinger model in Hamiltonian lattice theory, *Phys. Rev. D* **56**, 55–67 (1997).
- [36] T.M.R. Byrnes, P. Striganesh, R.J. Bursill, and C.J. Hamer, Density matrix renormalization group approach to the massive Schwinger model, *Phys. Rev. D* **66**, 013002 (2002).
- [37] H. Kröger and N. Scheu, The massive Schwinger model - a Hamiltonian lattice study in a fast moving frame, *Phys. Lett. B* **121**, 58–63 (1998).
- [38] T.M.R. Byrnes, C.J. Hamer, Z. Weihong, and S. Morrison, Application of Feynman-Kleinert approximants to the massive Schwinger model on a lattice, *Phys. Rev. D* **68**, 016002 (2003).
- [39] C. Adam, Improved vector and scalar masses in the massive Schwinger model, *Phys. Lett. B* **555**, 132–137 (2003).
- [40] E. Gardi, Why Padé approximants reduce the renormalization-scale dependence in QFT, *Phys. Rev. D* **56**, 68–79 (1997).
- [41] S.J. Brodsky, J. Ellis, E. Gardi, M. Karliner, and M.A. Samuel, Padé approximants, optimal renormalization scales, and momentum flow in Feynman diagrams, *Phys. Rev. D* **56**, 6980–6992 (1997).
- [42] G. Cvetič, Renormalization-scale-invariant continuation of truncated QCD (QED) series: an analysis beyond large- β_0 approximation, *Nucl. Phys. B* **517**, 506–520 (1998).
- [43] G. Cvetič, Improvement of the method of diagonal Padé approximants for perturbative series in gauge theory, *Phys. Rev. D* **57**, 3209–3213 (1998).
- [44] G. Cvetič, Applying generalized Padé approximants in analytic QCD models, *Phys. Rev. D* **84**, 056005 (2011).
- [45] G. Cvetič, Techniques and evaluation of QCD low-energy physical quantities with running coupling with infrared fixed point, *Phys. Rev. D* **89**, 036003 (2014).

Table Caption

Table 1: Ground-state energy of Schwinger model, for the varying dimensionless parameter $x = m/g$, in different approximations: Density matrix renormalization group, E_{DMRG} , Padé approximants, E_{PA} , fast moving frame estimates, E_{FMFE} , and self-similar root approximant E_5^* .

Table 1

x	E_{DMRG}	E_{PA}	E_{FMFE}	E_5^*
0.125	0.540	0.540	0.528	0.540
0.25	0.519	0.520	0.511	0.519
0.5	0.487	0.489	0.489	0.487
1	0.444	0.447	0.455	0.444
2	0.398	0.396	0.394	0.392
4	0.340	0.340	0.339	0.337
8	0.287	0.286	0.285	0.284
16	0.238	0.236	0.235	0.235

Ground-state energy of Schwinger model, for the varying dimensionless parameter $x = m/g$, in different approximations: Density matrix renormalization group, E_{DMRG} , Padé approximants, E_{PA} , fast moving frame estimates, E_{FMFE} , and self-similar root approximant E_5^* .